Ground-state configurations for Toom cellular automata: Experimental hints

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With computer simulations we investigate properties of stationary states of cellular automata with spins governed locally by the Toom majority rule; i.e., three neighbors, north, east, and center, vote for a next time step state of a center spin on a square lattice with periodic boundary conditions, as the candidates on ground states for the thermodynamic system arise. In particular, we compare the cluster development calculated according to the mean-field approach to experimental data, and carefully study critical phenomena similar to the phase transitions of the first and second types appearing in the model. [S1063-651X(97)03511-3]

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I. INTRODUCTION

Probabilistic cellular automata are discrete, usually nonreversible dynamic systems for which there exists an expectation that they can be included as objects of equilibrium statistical mechanics [1–4]. In general, the stationary distribution of trajectories for any stochastic model may be viewed as a canonical distribution under an effective Hamiltonian on the space of trajectories, in which each configuration interacts with its predecessor in time. The interaction energy in such a space is equal to the logarithm of the corresponding transition probability [5]. This general property applied to the cellular automata provides that the stationary measure μ arising on a configuration space $\mathcal{X}$ of cellular automata becomes equivalent to the equilibrium model of the so-called generalized Ising model in $(d+1)$ dimensions [1–3]. However, the parameters of this $(d+1)$-dimensional generalized Ising model are dependent on each other, which results in, for example, the free energy of the system being identically equal to 0 [6]. Moreover, the projection back to the d-dimensional system loses the notion of the Hamiltonian. Hence, the meaning of the so-called Gibbs measure, that is, the fundamental measure associated with every equilibrium statistical model [5,7], is lost also.

Therefore, establishing the relation between cellular automata systems and equilibrium systems is under constant investigation [8,4]. Not having the expressions for the interaction energy between objects responsible of the cellular automata behavior, one can try to search for relations to other fundamental notions of equilibrium statistical mechanics. Two of them, the phase diagram and the ergodicity [5,7], are of basic interest in the present paper. The phase diagram for any system provides division in the system parameter space into separate areas where the so-called pure phases exist or coexist. The stationary states of cellular automata can be considered as the pure phases of some thermodynamic system and so one can ask for the phase diagram in such a system. The ergodic properties of a cellular automata system are understood as the dependence of the stationary states on initial configurations [3]. By the stationary state of cellular automata we mean the state that is represented by a measure arising from the statistic properties obtained from averages over a great number of cellular automata stationary configurations.

In the present paper we concentrate on cellular automata acting locally under the majority rule, called the Toom rule [9,2]:

$$\sigma_{i,j}(t+1)=\mathrm{sgn}[\sigma_{i,j}(t)+\sigma_{i,j+1}(t)+\sigma_{i-1,j}(t)], \quad (1)$$

where $\mathcal{N}_{i,j}=[\sigma_{i,j}(t),\sigma_{i,j+1}(t),\sigma_{i-1,j}(t)]$ are center, east, and north nearest neighbor spins of the $x=(i,j)\in\mathcal{L}$ site on the square lattice, respectively.

The Toom automata dynamics, perturbed by random noise that mimics temperature effects, has been investigated rigorously by Lebowitz *et al.* [3]. At a high level of thermal noise, due to the fast decay of correlations between spins, the Gibbsianess of the Toom system has been proved rigorously.

For the low level or in the absence of noise this system is known to be nonergodic; i.e., the stationary state depends on the initial configuration [2,10]. The phase diagram in the space of the external magnetic field for these automata has been studied numerically by Bennett and Grinstein [2]. Their comparison of the Toom automata to the standard Ising model indicated that stationary states with the superfluous property existing within the Toom model are the source of the basic difference between the models compared.

The Gibbsian versus non-Gibbsian nature of the Toom system at low temperature is under continuous questioning [11,12]. Since the majority rules appear in the renormalization group transformations as the local rule for the so-called block spin interaction [7,13], properties of the Toom model are important to study.

The rigorous background of the zero-temperature thermodynamic investigations has been given by Pirogov and Sinai (see [5,14]). The idea of the Pirogov-Sinai theory consists in that the low-temperature diagrams are small deformations of the zero-temperature diagrams and the low-temperature limit Gibbs distributions describe only small local distortions of stable zero-temperature ground-state configurations. Therefore, after examining the phase diagram for deterministic Toom cellular automata we will investigate the influence of the thermal noise on this phase diagram.

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In the case of square cellular automata with periodic boundary conditions the general relation between properties of the local rule and configurations to which the evolution is attracted has been phenomenologically recognized [15]. It has been found that the stationary configurations arisen from the random initial configurations are those on which the local rule would establish globally a common shift of a state of the same neighbor. To identify these basic shift maps in Toom local rule let us specify, $\vartheta_0, \ldots, \vartheta_7$, the eight basic nearest neighbor states, called neighborhoods:

$$\vartheta_0=\begin{pmatrix} - & \\ - & - \end{pmatrix}, \quad \vartheta_1=\begin{pmatrix} + & \\ - & - \end{pmatrix},$$

$$\vartheta_2=\begin{pmatrix} - & \\ + & - \end{pmatrix}, \quad \vartheta_3=\begin{pmatrix} - & \\ - & + \end{pmatrix},$$

$$\vartheta_4=\begin{pmatrix} + & \\ + & - \end{pmatrix}, \quad \vartheta_5=\begin{pmatrix} + & \\ - & + \end{pmatrix},$$

$$\vartheta_6=\begin{pmatrix} - & \\ + & + \end{pmatrix}, \quad \vartheta_7=\begin{pmatrix} + & \\ + & + \end{pmatrix}. \tag{2}$$

The Toom rule (1) indicates three basic shift maps that are involved in the Toom dynamics: the center spin shift, the northern spin shift, and the eastern spin shift. Thus one can rewrite (1) in the three forms that correspond to these shifts:

$$\sigma_{(i,j)}(t+1)=\sigma_{i,j}(t)-2[\chi_{\vartheta_2}(\mathcal{N}_{(i,j)})+\chi_{\vartheta_5}(\mathcal{N}_{(i,j)})],$$

$$_{(i,j)}(t+1)=\sigma_{i-1,j}(t)-2[\chi_{\vartheta_1}(\mathcal{N}_{(i,j)})+\chi_{\vartheta_6}(\mathcal{N}_{(i,j)})], \tag{3}$$

$$\sigma_{(i,j)}(t+1)=\sigma_{i,j+1}(t)-2[\chi_{\vartheta_3}(\mathcal{N}_{(i,j)})+\chi_{\vartheta_4}(\mathcal{N}_{(i,j)})],$$

with χ_{ϑ_k} denoting the characteristic function of the ϑ_k neighborhood state. Namely, if $\vartheta_k=(N_k, C_k, E_k)$ then

$$\chi_{\vartheta_k}(\mathcal{N}_{(i,j)})=\tfrac{1}{8}(N_k+\sigma_{(i-1,j)})(C_k+\sigma_{(i,j)})(E_k+\sigma_{(i,j+1)}).$$

Let us note that the three shifts listed in Eq. (3) are equivalent to each other in the global shift adjusting process because they point to the same number of neighborhoods—i.e., two neighborhoods, which do not obey the given shift operation. Hence, the maximal list of so-called excluded blocks must contain all six neighborhood states: $\vartheta_1, \ldots, \vartheta_6$ and, therefore, the stationary configurations resulting from the Toom rule will be built of the remaining two neighborhoods.

However, the thermodynamic properties of the Toom model such as ergodicity or critical changes in stationary states cannot be explained within the geometric-dynamic approach presented above.

This paper is organized in the following way: First, in Sec. II, we indicate the discrepancy between the results arising from the statistical approach via the mean field methods proposed by Gutowitz *et al.* [16,17] adapted to suitably study the neighborhood development [18,19], and the results of simulations. The failure of the standard cellular automata statistical methods becomes the motivation for our further studies. In Sec. III we present results of examinations of the ergodic properties in the Toom deterministic model, i.e., without noise, on the square lattice imposed with periodic boundary conditions. The influence of the boundary condition is tested to extract properties that depend on the finite lattice size. Then, the set of zero-temperature ground-state configurations is specified. We close this section with the propositions that characterize both the set of ground states and the conditions under which the system is led to the particular ground state. This latter proposition results from the so-called damage spreading investigations [20–22] that allow one to examine the basin boundaries between attracting stationary configurations. Section IV is aimed at verification of the stability of the ground-state configurations with respect to the stochastic perturbation; i.e., the probabilistic Toom cellular automata are simulated. The critical dependence of the stationary state on the noise is tested with the standard Monte Carlo method [23,24] and scaling critical exponents are extracted to characterize the phase transition arisen. In the last section we comment on the crucial observations.

II. STATISTICAL INVESTIGATIONS

The basic idea of the statistical methods leading to the stationary measure consists in the application of the famous Kolmogorov consistency theorem. This theorem assures that under the so-called self-consistency condition, the set of additive and positive function defined on the cylinder sets of a given configuration space extends uniquely to a measure defined on the whole configuration space. Therefore the suitable assignment of probabilities to the finite configurations of a lattice (cylinder sets here are usually called blocks) can lead to the proper probability measure (called the block measure). However, the real problem occurs in arranging different block measures into a sequence that in the limit will tend to the stationary measure adjusted to the given dynamic process.

Thus, to apply statistical tools to cellular automata one needs to consider the proper hierarchy of self-contained blocks, which are consistent with each other and for which the transition probabilities between the layers in this hierarchy are closely related to cellular automata dynamics.

For our investigations we choose blocks built on the base of the Toom neighborhood (2). This means that we consider the hierarchy of blocks:

(1) $\mathcal{B}_0$, one spin blocks: (a).

(2) $\mathcal{B}_1$, blocks of nearest neighbors:

$$\begin{pmatrix} b & \\ a & c \end{pmatrix}.$$

(3) $\mathcal{B}_2$, blocks of next-to-nearest neighbor blocks:

$$\begin{pmatrix} d & & \\ b & e & \\ a & c & f \end{pmatrix}.$$

(4) …., .

It is easy to see that the blocks of this family do not give a unique cover to other finite blocks possible on the lattice. Therefore, the blocks introduced cannot satisfy the self-consistency condition. However, the blocks from different layers of this hierarchy are consistent with each other via dynamic relations. This last feature allows one to study the development of correlations in the system and, moreover, to observe the blocks from the excluded list.

Let us concentrate on the layer of $\mathcal{B}_1$ blocks. The standard method based on the Frobenius-Perron operator [18,19] applied to these blocks provides the link between the block probabilities in subsequential time steps as

$$P_{n+1}\begin{pmatrix} b' & \\ a' & c' \end{pmatrix} = \sum_{\begin{pmatrix} d & & \\ b & e & \\ a & c & f\end{pmatrix}:\; b'=T\binom{d}{be},\; a'=T\binom{b}{ac},\; c'=T\binom{e}{cf}} P_n \begin{pmatrix} d & & \\ b & e & \\ a & c & f \end{pmatrix},$$

where T denotes the Toom rule. The probability for a parent neighborhood block to be expressed by conditional probabilities of some smaller blocks is

$$P\begin{pmatrix} d & & \\ b & e & \\ a & c & f \end{pmatrix} = P\left(\begin{matrix} e & \\ c & f \end{matrix}\;\middle|\;\begin{matrix} d & \\ b & e \\ a & c \end{matrix}\right) \times P\left(\begin{matrix} d & \\ b & e \end{matrix}\;\middle|\;\begin{matrix} b & \\ a & c \end{matrix}\right) P\begin{pmatrix} b & \\ a & c \end{pmatrix}.$$

Assuming that the conditional probabilities in the above formula depend only on spins that belong to the common part (the standard assumption, called maximal entropy assumption [17]) one obtains the approximate relation

$$P\begin{pmatrix} d & & \\ b & e & \\ a & c & f \end{pmatrix} \approx P\begin{pmatrix} b & \\ a & c \end{pmatrix} P\left(\begin{matrix} d & \\ b & e \end{matrix}\;\middle|\; b\right) \times P\left(\begin{matrix} e & \\ c & f \end{matrix}\;\middle|\;\begin{matrix} e \\ c \end{matrix}\right), \tag{5}$$

that is, closed with respect to $\mathcal{B}_1$ blocks and, therefore, easy to use for the iteration procedure.

We iterate the formula (4) with the simplification (5) assuming that initially spin states are set at random with the probability $p=0.51$ for the $+$ state. On the other hand, we perform a computer simulation starting the evolution with typical initial configurations prepared by a random toss at the probability $p=0.51$ for the $+$ state. At each time step the distribution of neighborhood states from the list (2) is calculated. In Fig. 1, we plot the results of the successive iterations of the $\mathcal{B}_1$ block probabilities and the averages of the results obtained in computer experiments. The discrepancy between these curves is noticeable after just a few steps. The maximum entropy assumption, which randomly scatters blocks all over the configuration, breaks the development of correlations between spins, and therefore changes the lattice configuration faster than the simulation provides.

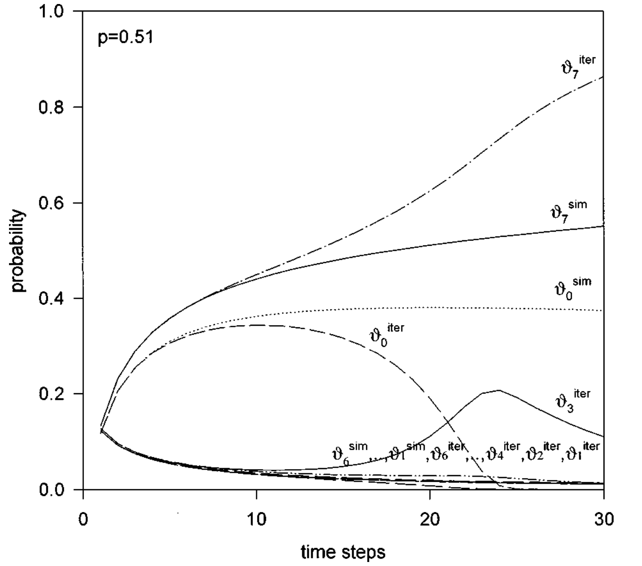

FIG. 1. Distribution of neighborhoods: $\vartheta_0, \ldots, \vartheta_7$ obtained by simulation (denoted ϑ^{sim}) and by iteration (denoted $\vartheta^{\mathrm{iter}}$). The special shape of $\vartheta_3^{\mathrm{iter}}$ is induced by the applied maximal entropy assumption.

One can hope that by introducing larger blocks these discrepancies would vanish. However, it is easy to observe that larger blocks only cause greater complication to a few initial steps of the iteration. Blocks of any size will soon appear too small to reconstruct growing with time dependencies between spins, because of the steady presence of the adjusting spin states process. This process is due to the special property of Toom cellular automata called the *eroder property* [9]. This property implies that any finite island made of one spin state disappears at finite time steps.

In the Toom cellular automata this property is effected by the generic ability of this system to produce moving *flat interfaces*. The flat interface is a straight line on a lattice configuration that separates the phase of $(+)$ from the phase of $(-)$ [7]:

$$\begin{matrix} + & + & + & - & - & - \\ + & + & + & - & - & - \\ + & + & + & - & - & - \\ + & + & + & - & - & - \\ + & + & + & - & - & - \end{matrix}$$

stable flat interface

$$\begin{matrix} - & + & + & + & + & + \\ - & - & + & + & + & + \\ - & - & - & + & + & + \\ - & - & - & - & + & + \\ - & - & - & - & - & + \end{matrix}$$

moving flat interface. (6)

When one considers the Toom model on a triangular lattice [1,3] then all flat interfaces are mobile. The Toom model considered on a square lattice (sometimes this system is called the NEC system [9,2]) possesses two stable flat interfaces, horizontal and vertical, and the third one is diagonal moving southwestward regardless of which phase is on which side of the interface. This free propagation of the flat interface is supposed to be responsible for interaction of any distant spins. The flat-interface configurations, also called *superfluous* configurations, can be considered as the proposition of distinct thermodynamic phases from the original Ising homogeneous phases: $(+)$ and $(-)$.

III. SET OF ZERO-TEMPERATURE CONFIGURATIONS

To fix the set of zero-temperature configurations we perform a computer experiment: Beginning with typical initial configurations, i.e., configurations where spin states are randomly set with some fixed probability p to toss the $+$ state on the square lattice $L\times L$ with periodic boundary conditions, we observe the time development of cellular automata with (1) local rule up to $5L$ time steps. In Figs. 2(a)–2(e) we collect results of these experiments together with the analysis of the obtained results.

Figure 2(a) presents the average time $\langle T\rangle$ by means the number of evolution steps that is needed by the system to reach stabilization. These results provide both the lattice size L and the initial state characterization p influence. The average time $\langle T\rangle$ normalized by the number of spins to adjust, i.e., divided by L^2, indicates the exponential dependence on p when $p\in(0,4)$ and the polynomial dependence on L within the same p interval; see Fig. 2(b). Our estimations lead to the formula

$$\langle T(L,p)\rangle = L^{2-1.82e^{-0.156p}} e^{4.98pL^{0.045}} \quad \text{for} \quad p<0.4. \tag{7}$$

If p is approaching 1/2, the increase in time steps to reach stabilization depends on the lattice size differently; see Fig. 2(c). This is the p area where the Toom cellular automata lose ergodicity. For each lattice size there exists a p interval around $p=1/2$, where the stationary states of different types can occur [10]. Figure 2(d) presents the distribution of final configurations if initial states are random with $p=1/2$ and for different lattice sizes. We note that with the growth of the lattice size, the probability of obtaining the stationary state with the flat interface stabilizes at the value 1/3 independently of the lattice dimension [the more complicated patterns like zigzag configurations have been observed only a few times: $L=48$ (once), $L=384$ (twice), $L=500$ (twice)].

Comparing the development of the cluster structures in distinct lattices by size we test the finite lattice size influence. Figure 2(e) shows the probability of finding a spin in the *down* state on a configuration at the fixed moments of time: $t=48,\ldots,500$ and for different lattice sizes $L=48,\ldots,500$ but keeping always $t\leqslant L$.

The above results lead to the following observations:

(1) The periodic boundary conditions break the free development of randomly scattered homogeneous islands — clusters; the process of adjusting these clusters to the periodic boundary conditions increases the time to stabilization to approximately $2L$.

(2) The final stationary configurations for the Toom model with periodic boundary conditions can be grouped into the following classes: homogeneous with all spins *up*; homogeneous with all spins *down*; *flat-interface* configurations; *zigzag* configurations— there are two separated areas with different *flat-interface* configurations on a final pattern.

On the other hand, the idea explained in the Introduction of the local shifts' competition that drives the cellular automata evolution to a configuration on which the common global shift can be performed indicates the close relation between the geometric properties of the obtained stationary patterns and the geometric properties of the lattice. Therefore, we perform the following additional experiments: we violate the regularity of the boundary horizontal condition by introducing the length of rows at random. The results obtained in these experiments provide the stationary configurations classification: homogeneous pattern with all spins *up*; homogeneous pattern with all spins *down*; patterns with two horizontal stripes; patterns with four horizontal stripes, etc. Hence, both the vertical and diagonal flat interface configurations are not observed.

Therefore, let us conclude our results.

Proposition 1:

(A) The list of possible candidates for ground states in the Toom model on the square lattice is determined by the boundary conditions assigned to the system.

(B) The periodic boundary conditions fix the set of ground states to

$$\mathcal{X}_0 = \{\{\text{all}(+)\},\{\text{all}(-)\},\ \{\text{flat-interface}\}\}. \tag{8}$$

The damage spreading study [20,21] of the area where Toom cellular automata with periodic boundary conditions loses ergodicity has provided the fractal structure of the basin boundary between the attracting ground states [22]. Moreover, the detailed observation of the distance between the configurations that are attracted to the different stationary configurations results in the following proposition.

Proposition 2:

(A) If an initial configuration σ_0 is led to any flat-interface configuration,

$$\sigma_0 \to_{t\to\infty} (\text{flat interface}),$$

then there always exists at least one such initial configuration σ distinct from σ_0 by one spin state such that spins

$$\sigma \to_{t\to\infty} \begin{cases} \text{all spins up} \\ \text{other flat interface (same type of stripes as } \sigma_0). \\ \text{all spins down} \end{cases}$$

(B) If an initial configuration σ_0 is led to any homogeneous configuration, i.e.,

$$\sigma_0 \to_{t\to\infty} \begin{cases} \text{all spins up} \\ \text{all spins down} \end{cases}$$

and if there exists σ distinct from σ_0 by one spin state such that σ is led to the different stationary configuration from σ_0, then it is always that

$$\sigma \to_{t\to\infty} \text{flat interface (all types)}.$$

Hence, the typical initial configurations that led to the *flat-interface* stationary configurations form a dense subset in

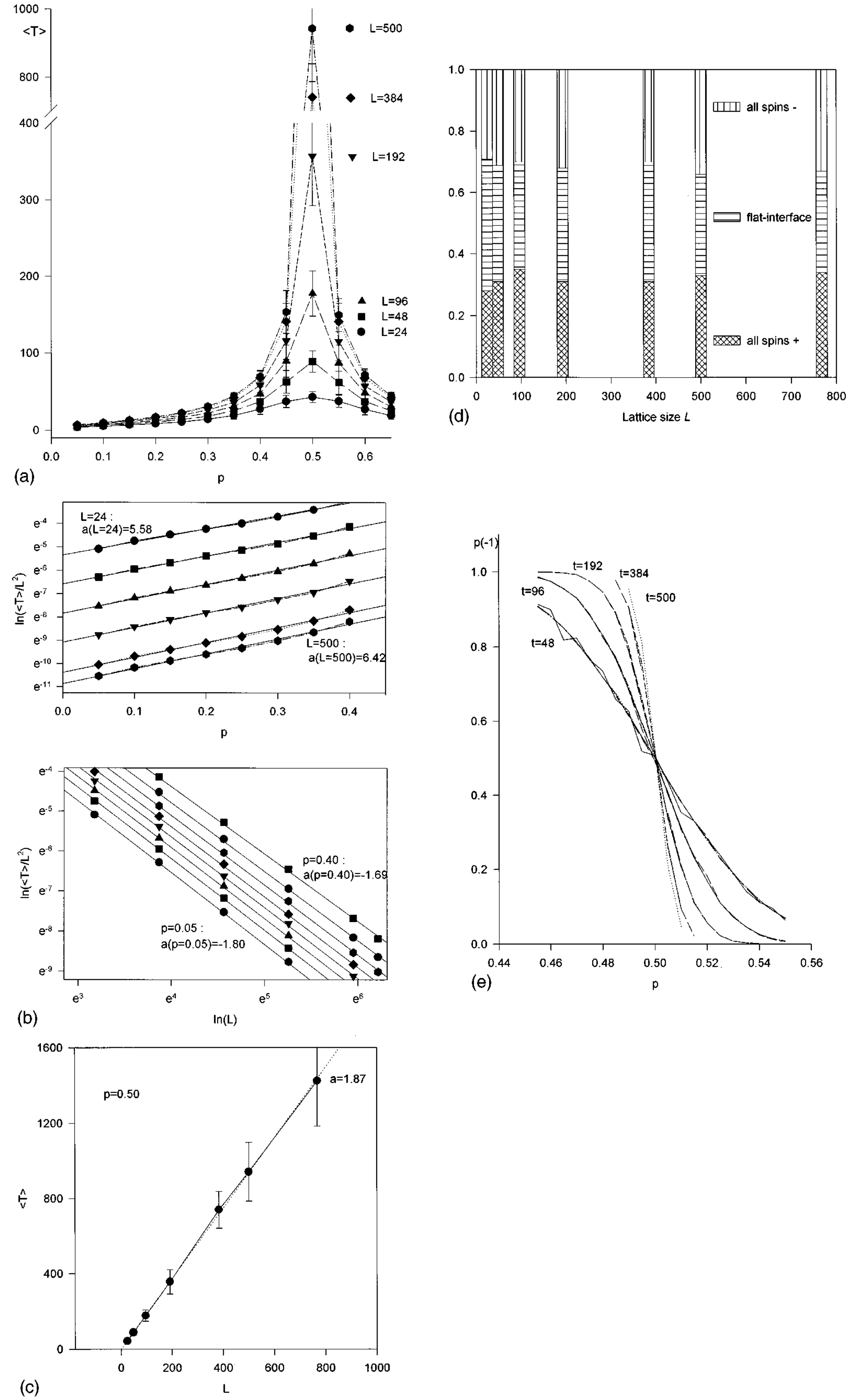

FIG. 2. (a) Mean time to reach the stabilization with respect to p of an initial state and L lattice size. The averages are made after 300 experiments for each lattice size: $L=24,48,96,192,384,500$. The STD error for these results are included. (b) ln plot of the mean time of adjusting a single spin state for the lattice size $L=24,48,96,192,384,500$ vs p (the top plot) and ln-ln plot of the mean time of adjusting a single spin state for different p of initial typical configuration p:0.05,0.1,0.15,0.2,0.25,0.3,0.35,0.4, vs lattice size L (the bottom plot). Values denoted by a correspond to the linear regression for the particular curves. (c) Maximal mean time for Toom system to reach the stabilization vs lattice size L. Initial configurations are random with $p=1/2$. The parameter a provides the linear regression for the data. (d) Distribution of final configurations among the three main classes: all spins *up*, all spins *down*, and *flat-interface* configurations vs lattice size L. (e) Probability of finding a spin in the *down* state on lattices with different sizes: $L=24,48,96,192,384,500$ and at fixed time steps. The moments of observation are chosen to be less than or equal to the given lattice size.

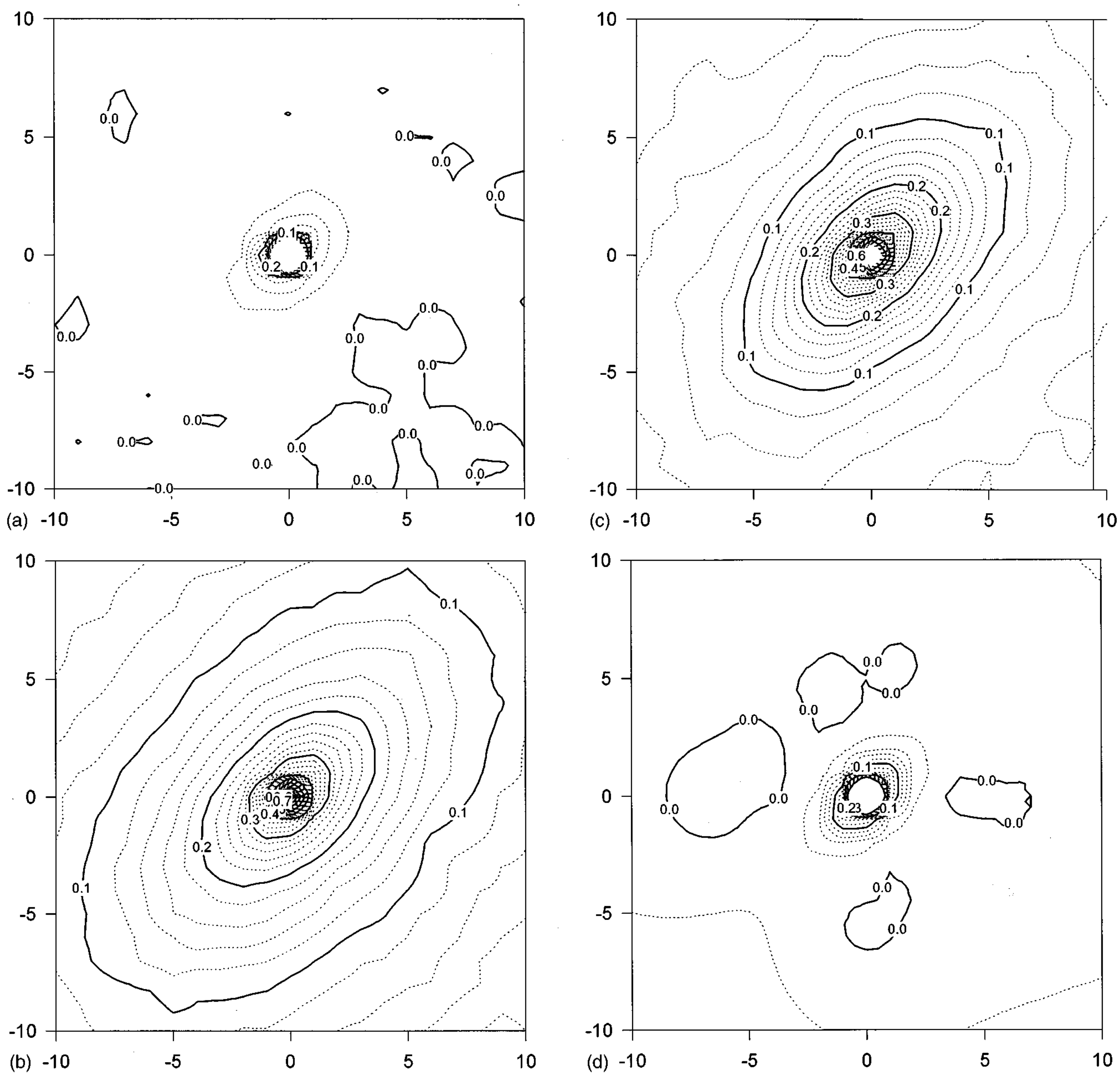

FIG. 3. Two-point correlations between spins from the stationary configurations vs lattice sites for different levels of the error ε : (a) $\varepsilon = 0.08$, (b) $\varepsilon = 0.09$, (c) $\varepsilon = 0.10$, (d) $\varepsilon = 0.19$. Results are presented as the contour graphs. The point (0,0) means the center of a lattice. The dotted lines divide the major intervals into 5 equal parts.

the space of initial random configurations at the basin boundaries of both attracting homogeneous configurations in the Hamming distance topology [21].

IV. GROUND-STATE PROPERTIES

Ground states of the thermodynamic lattice system where sites are occupied by a discrete systems with a finite state space are known to satisfy the so-called Peierls's condition [5,7,14]. This condition states that there exists an essential energy gap between any ground state and other states. This gap makes the ground states stable against the small perturbation. In the subsequent experiments we test the stability of ground states arising from Toom cellular automata. Beginning the evolution with the ground state, we observe changes appearing on the stationary states caused by the thermal noise. The thermal noise is introduced to the deterministic dynamics (1) by the standard ε error [3], i.e., by setting the probability ε that the system locally acts against the Toom rule. The changes observed are represented as the two point correlation function of spin states $\mathrm{corr}(x,y)$.

Our results for homogeneous $(+)$ initial configuration are collected in the series of plots, Fig. 3, presenting data versus different error levels ε : $0.08,\ldots,0.19$.

It occurs that there are three areas of stationary configuration dependence on the level of the noise ε:

— $\varepsilon < 0.08$: The correlation function takes values significantly distinct from zero for the four nearest neighbors. Spins separated by more than one lattice unit are independent. The total magnetization of the stationary configuration is greater than 0.63, which means that the probability of finding a spin in *up* state is $p>0.9$.

— $0.08<\varepsilon<0.20$: The rapid change in the stationary configuration properties appears when the noise parameter crosses 0.08. The correlations between spins separated by up to twenty units appear. Notice that dependencies between spins reflect the geometry of Toom dynamics—the correlation between spins lying along the diagonal SW-NE is at least two times stronger than that between spins on the other diagonal. We assume that with crossing the value of 0.08 with ε the equivalent development of homogeneous clusters of both types begins and, therefore, one can say that the system is about at the phase transition of the second type. The Gibbsian nature of the stationary states appearing at this critical interval by studying the locality of the interactions arisen has been carefully considered in [12]. The finite lattice size studies based on the fourth order magnetization cumulant dependence on the lattice size have provided $\varepsilon_{cr} \approx 0.091\pm0.002$ for the Toom system in the thermodynamic limit. The critical parameter responsible for the finite lattice size effect ν has been found, $\nu=0.90\pm0.02$ [12]. In the present, we estimate the remaining critical parameters to characterize completely the scaling properties of the Toom system in a neighborhood of ε_{cr}. For equilibrium thermodynamic systems these properties are supposed to be independent of local interactions [5,23] and therefore determine the universality class of a considered system.

The first parameter β results from the decay of the mean magnetization per spin $\langle m\rangle$ when $\varepsilon\nearrow\varepsilon_{cr}$, i.e., $\langle m\rangle \propto (\varepsilon_{cr}-\varepsilon)^{\beta}$ for $\varepsilon<\varepsilon_{cr}$; see Fig. 4(a). The second parameter γ arises from dependence on temperature of the susceptibility χ and it can be extracted from the decay of fluctuations of the mean magnetization per spin after crossing ε_{cr} (when $\langle m\rangle\approx 0$), i.e., $\chi\propto\langle m^2\rangle\propto(\varepsilon-\varepsilon_{cr})^{-\gamma}$ for $\varepsilon\searrow\varepsilon_{cr}$; see Fig. 4(b).

The standard scaling theory [23] states that these two critical parameters, $\beta\approx 0.99$ and $\gamma\approx 0.45$, determine the values of the two remaining critical parameters α and δ in the following way: $\alpha=2-\gamma-2\beta\approx-0.45$ and $\delta=1+\gamma/\beta \approx 1.45$, which yields the fractal dimension for this phase transition $d_f\approx 1.18$. The surprising (because it is negative) value of α scales the unknown function (the analogs of the thermodynamic free energy function) at $|\varepsilon-\varepsilon_{cr}|\rightarrow 0$ with the power $2+0.45$.

— $\varepsilon>0.20$: The length of observed dependencies between spins decays but it decays so slowly that it is difficult to point to the sharp limit of this ε interval. However, when the error level exceeds 0.20 since the magnetization of the whole configuration is zero and the correlations are zero for spins separated further than one lattice unit, we can consider these stationary configurations as random. Namely, the corresponding stationary measure is of the Bernoulli type with $p=1/2$.

The analogous simulations are performed when the initial configuration is of the *flat-interface* type. It appears that in the case of $\varepsilon<0.08$, if the thermalization time is left long enough, e.g., longer than 100 000 time steps, then the flat-interface structures completely disappear. Either the $(+)$ or the $(-)$ phase remains on the lattice. To learn more about this phenomenon let us consider the role played by the spins belonging to the interface between the homogeneous phases

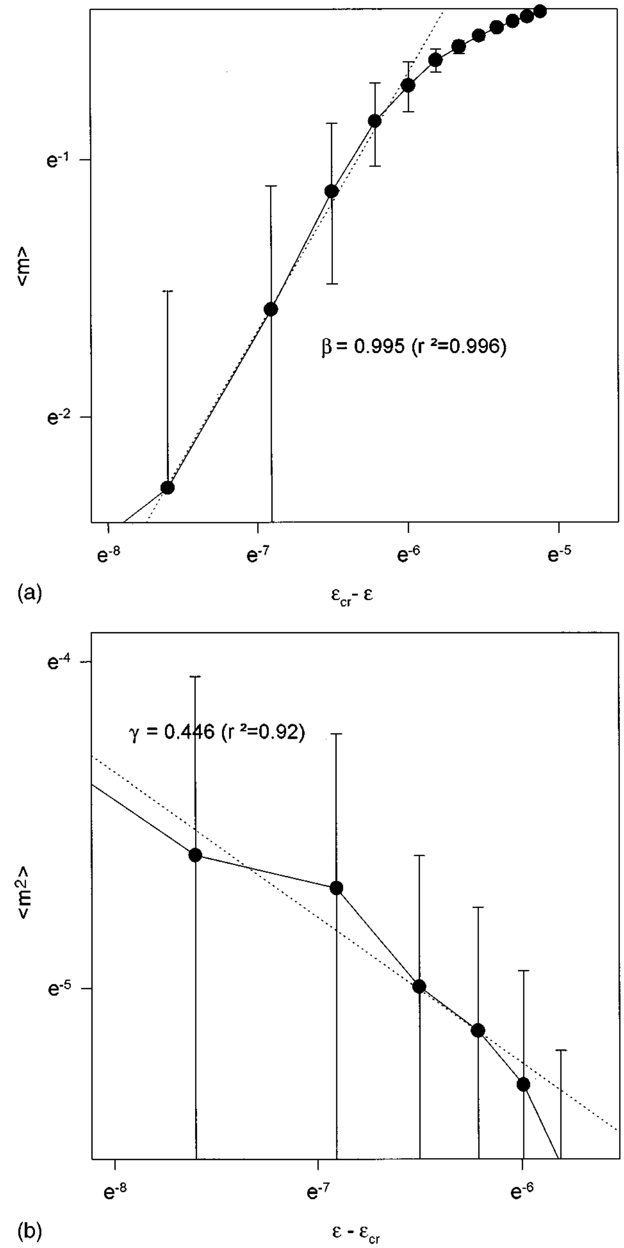

FIG. 4. (a) ln-ln plot of the mean magnetization per spin $\langle m\rangle$ vs $\varepsilon_{cr}-\varepsilon$ for $\varepsilon_{cr}=0.091$, $L=200$. The dotted line represents the linear regression with the given β slope. (b) The ln-ln plot of the fluctuation of magnetization per spin $\langle m^2\rangle$ vs $\varepsilon-\varepsilon_{cr}$ for $\varepsilon_{cr}=0.091$, $L=200$. The dotted line represents the linear regression with the given γ slope.

in the flat-interface configuration [see Eq. (6)].

It is easy to notice that any flip of a spin from the left side of the vertical interface produces not decaying but propagating perturbation. Hence, if the temperature error occurs at the interface, then this single change will propagate until it converts back the perturbed interface into the flat interface. It takes L time steps. There are $2L$ sensitive in this sense spin sites in any stable flat-interface configuration on a square lattice with periodic boundary conditions. This implies that the probability that the error occurs within these spins is equal to $2\varepsilon L$ at each time step. In particular, for $\varepsilon=0.005$ and $L=100$, this probability reaches 1. The propagation of the error causes the entire configuration to look like a pattern with a moving interface. Eventually, after 100 000 steps, each interface is about to be moved by 50 000 lattice units. Since each of the two interfaces moves at own speed, soon the whole configuration is transferred into the homogeneous phase. However, which pure phase wins this ''speed'' competition seems to occur at random.

If the value of the error ε increases, then together with the interface change the ''volume'' effects appear as in the homogeneous configurations. Therefore, the properties of the system are completely the same as in the homogeneous case.

Let us summarize our experiments.

Proposition:

For the Toom probabilistic cellular automata imposed with the periodic boundary conditions we have the following:

(A) The homogeneous ground-state configurations are stable against random perturbation less than 0.09, so that the homogeneous configurations can be considered rigid ground-state configurations [7]. The rigidity of these ground states with increasing level of ε is lost— the system undergoes the phase transition of the second type.

(B) The flat-interface configurations are not stable against random perturbation. There exist lattice sites that are sensitive to a single spin perturbation in the sense that the perturbation of this spin state propagates freely, causing the interface between homogeneous phases to disappear.

V. CONCLUSIONS

Advances in computer technology allow one to design more complex experiments and thus to obtain different insights into the old problem. Stimulated by the McIntosh idea [25] that, thanks to both growing understanding of the subject and better machines, it is time to return to the very first topic in the cellular automata general theory, which consists in getting good probability measures to describe automata and their evolution, in the present paper we have repeated some of the old simulations [10], however in a fashion that has guaranteed getting distinct information:

(i) We have failed with the results after applying the standard mean field methods that would provide the stationary measure for Toom cellular automata from the block distribution properties.

(ii) The nonergodic behavior of the system initiated on random configurations has been investigated with methods of dynamic systems. In particular, we have observed the stationary configuration development with respect to the Bernoulli parameter p of the initial random configuration and the influence of both the finite lattice size and the geometry of boundaries. On the other hand, the examination of the basin boundary between attracting configurations has pointed out the very sensitive structure of this basin boundary in the space of the initial configurations. Such a structure is characteristic for chaotic systems [23,26]. We can conclude that periodic boundary conditions lead to the chaotic behavior of the Toom system in an area where the phenomenon of the phase transition of the first order occurs.

(iii) We have found the scaling exponents from the behavior of the mean magnetization in the stationary states near the critical point in the same way that the scaling theory deals with equilibrium thermodynamic systems to describe the phase transition of the second type. We have found a qualitativy different role of the flat-interface stationary states from the role of the homogeneous stationary states. Although the flat-interface states do exist in the zero-temperature phase diagram they do not enter the nonzero-temperature phase diagram. The presence of the flat-interface states is caused by periodic boundary conditions rather than the effect of interactions arising from Toom local rule. Moreover, in the standard kinetic Ising model, the temperature effects are strongly moderated by interactions, namely, the noise acts differently on spins belonging to clusters from other spins (see, e.g., [21]). In the Toom local dynamics there is no such a distinction. Therefore, the magnetization is lost more quickly (with $\beta\approx0.99$) than the magnetization in the Ising model ($\beta_{\rm Ising}=0.125$). However, since the obtained value of β is smaller than 1, then some effects of Ising-type interactions such as preferences for one spin state clusters must be present in the Toom cellular automata. The weak decay of the fluctuation that is manifested by low value of γ (for Ising interactions $\gamma_{\rm Ising}=1.75$) results from the superfluous property of the diagonal flat-interface configurations.

Finally, sometimes the noise effect is introduced to the cellular automata system as the probability that locally the rule does not obey the deterministic rule, which denotes that with probability ε the results of the rule are random. However, the random choice of a spin state denotes that with probability $\frac{1}{2}\varepsilon$ the state will agree with the deterministic rule result and with probability $\frac{1}{2}\varepsilon$ the state opposes the rule. Hence, such a system performs the deterministic rule with $1-\frac{1}{2}\varepsilon$ and acts oppositely with probability $\frac{1}{2}\varepsilon$. This influences the $\varepsilon_{\rm cr}$ value but does not influence the scaling properties.

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